AD-A128 072 APPROXIMATION BY SMOOTH BIVARIATE SPLINES ON A THREE-DIRECTION MESH(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER R JIA MAR 83 MRC-TSR-2494 1/1

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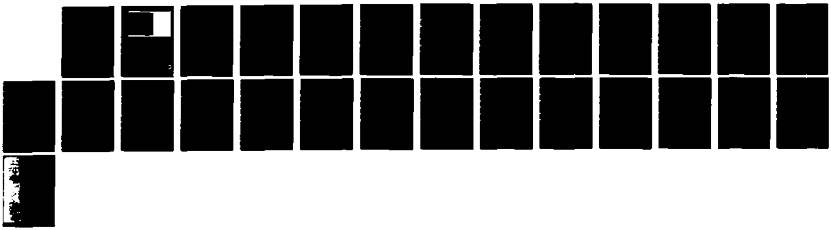

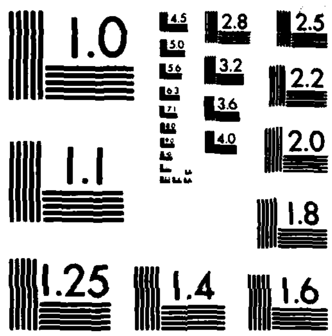

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MRC Technical Summary Report # 2494

APPROXIMATION BY SMOOTH BIVARIATE SPLINES ON A THREE-DIRECTION MESH

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March 1983

(Received February 15, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

83 05 06-132

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

Let $S := \pi^{\rho}_{k,\Delta}$ be the space of bivariate piecewise polynomial functions in C^{ρ}, of degree $\leq k$, on the mesh Δ obtained from a uniform square mesh by drawing in the same diagonal in each square.

de Boor and Höllig have given the following upper bound

$$m \leq m(k) := \min\{2(k-\rho),\ k+1\}$$

for the approximation order m of S.

In this paper, the lower bound

$$m \geq m(k) - 2$$

is demonstrated. This result is close to de Boor and Höllig's conjecture that m never differs from $m(k)$ by more than 1.

Incidentally, the approximation order of $\pi^{1}_{4,\Delta}$ is shown to be 4.

AMS (MOS) Subject Classifications: 41A15, 41A63, 41A25.

Key Words: B-splines, bivariate, degree of approximation, pp, quasi-interpolants, linear functionals, smooth, spline functions.

Work Unit Number 3 - Numerical Analysis

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

Univariate splines have been proved quite useful in practice. However, if one wants to fit a surface, or solve a partial differential equation numerically, one would naturally think of using multivariate splines. Here splines still mean piecewise polynomial functions. In this respect, a basic question is to ascertain, for a given mesh Δ and a family S of splines on Δ, what its optimal approximation order is. This question is challenging even for a regular triangular mesh Δ, as soon as one demands that the approximating functions have a certain amount of smoothness. The report records a step toward answering the above question.

APPROXIMATION BY SMOOTH BIVARIATE SPLINES
ON A THREE-DIRECTION MESH

Rong-qing Jia

1. Introduction

In this paper we study approximation order of smooth bivariate splines on a three-direction mesh. The work in this respect was initiated by de Boor and DeVore [BD] and de Boor and Höllig [BH 1,2,3]. Here we follow them and introduce some notations. Let

$$\Delta := \bigcup_{n\in\mathbb{Z}} \{x \in \mathbb{R}^2;\ x(1) = n,\ x(2) = n,\quad \text{or}\quad x(2)-x(1) = n\} \quad .$$

Namely, the mesh Δ is obtained from a uniform square mesh by drawing in the same diagonal in each square. Let

$$S := \pi^{\rho}_{k,\Delta} := \pi_{k,\Delta} \cap C^{\rho}$$

be the space of bivariate pp (piecewise polynomial) functions in C^{ρ}, of total degree $\le k$, on the mesh Δ. Also, by π_k we denote the space of polynomials of total degree $\le k$. We are interested in the approximation order of S. The approximation order of S is, by definition, the integer m for which the following holds: For all sufficiently smooth function f,

$$\operatorname{dist}(f,S_h) = O(h^m)$$

while, for some C^{∞}-function f,

$$\operatorname{dist}(f,S_h) \ne o(h^m) \quad .$$

Here, the scale (S_h) of approximating spaces is generated from S by simple scaling,

$$S_h := \sigma_h(S)$$

with

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$$(\sigma_h f)(x) := f(x/h), \quad \text{all} \quad f,\ x,\ h \quad .$$

de Boor and DeVore have given the following lower bound for m (see [BD]):

$$m \geq \rho+2 \quad \text{in case} \quad \rho \leq \rho(k) := \lfloor (2k-2)/3 \rfloor \quad .$$

In contrast, S has approximation order 0 for $\rho > \rho(k)$.

An upper bound for m has been obtained by de Boor and Höllig (see [BH 3; Theorem 3]:

$$m \leq m(k) := \min\{2(k-\rho),\ k+1\} \quad .$$

de Boor and Höllig also show that the approximation order of $\pi^1_{3,\Delta}$ is 3 rather than 4 (see [BH 2]). Thus the approximation order of S may differ from $m(k)$ by 1. Based on those investigations, de Boor and Höllig raised the following

<u>Conjecture</u> ([BH 3]). The approximation order of $S = \pi^\rho_{k,\Delta}$ never differs from its upper bound $m(k)$ by more than 1.

In this paper, we shall show that the approximation order of $S = \pi^\rho_{k,\Delta}$ never differs from $m(k)$ by more than 2. The proof of this result will be based on a quasi-interpolant scheme. For the record we state the following

<u>Theorem 1</u>. <u>Suppose that</u> $B \in S$ <u>with</u> supp B <u>finite</u>. <u>If the map</u>

$$T : p \mapsto \sum_{z \in \mathbb{Z}^2} p(j)B(\ -j)$$

<u>is one-to-one and onto</u> π_n, <u>then</u>

$$\operatorname{dist}(f, S_h) = O(h^{n+1})$$

<u>for all sufficiently smooth functions</u> f.

The argument in [BH 1; Section 6] essentially gives the proof for Theorem 1. We do not need to repeat the proof here.

To construct an element $B \in S$ with the property required by Theorem 1, we shall emply box splines, which were introduced by [BD] and [BH 1]. In section 2, we develop some preliminary results from univariate B-spline

theory. In section 3, we elaborate some properties of box splines on the three-direction mesh Δ. In section 4, we construct an element $B \in S$ with the property required by Theorem 1, and therefore prove our main results. In section 5, we show that the approximation order of $\pi^1_{4,\Delta}$ is 4. This illustrates that the approximation order of $\pi^\rho_{k,\Delta}$ might be exactly k when $k = 2\rho+2$.

2. Some preliminary results from univariate B-spline theory.

Let $\underline{t} = (t_i)_{-\infty}^{\infty}$ be a knot sequence. Recall that

$$M_{i,k}(x) := k[t_i,\ldots,t_{i+k}](\cdot-x)_+^{k-1}$$

is a normalized B-spline of order k for each $i \in \mathbb{Z}$. Also we write

$$N_{i,k}(x) := (t_{i+k} - t_i)M_{i,k}(x)/k \quad .$$

If p is a polynomial of degree $< k$, then

$$p = \sum_{i\in\mathbb{Z}} (\lambda_i p)N_{i,k} \quad ,$$

where λ_i is the linear functional defined by

$$\lambda_{i,k}f := \sum_{j<k} (-)^{k-1-j}\psi_{i,k}^{(k-1-j)}f^{(j)}(\tau_i)$$

with

$$\psi_{i,k}(x) := (t_{i+1}-x) \cdots (t_{i+k-1}-x)/(k-1)!$$

and $\tau_i \in (t_i,t_{i+k})$ (see [BF]; also [B]). Now suppose $t_i = i$, all $i \in \mathbb{Z}$. Then $N_{i,k} = M_{i,k}$ and

$$\psi_{i,k}(x) = (i+1-x) \cdots (i+k-1-x)/(k-1)! \quad .$$

It is easily seen that there exist unique constants $a_{\ell,k-1}(\ell=0,1,\ldots,k-2)$ such that

$$\psi'_{0,k}(x) = \sum_{\ell=0}^{k-2} -a_{\ell,k-1}(\ell+1-x) \cdots (\ell+k-2-x)/(k-2)! \tag{1}$$

Comparing the coefficient of x^{k-2} on both sides of (1), we obtain

$$\sum_{\ell=0}^{k-2} a_{\ell,k-1} = 1 \quad . \tag{2}$$

If f is a polynomial of degree $\le k-2$, then

$$p := \sum_{i\in\mathbb{Z}} f(i)N_{i,k-1}$$

is also a polynomial of degree $\le k-2$. On the one hand,

$$f(i) = \lambda_{i,k-1}p \quad .$$

On the other hand,

$$p = \sum_{i\in\mathbb{Z}} (\lambda_{i,k}p)N_{i,k} \quad .$$

Pick $\tau_i \in (i+k-2, i+k-1)$ and calculate $\lambda_{i,k}p$ as follows:

$$\begin{aligned}
\lambda_{i,k}p &= \sum_{j<k} (-)^{k-1-j}\psi_{i,k}^{(k-1-j)}(\tau_i)p^{(j)}(\tau_i) \\
&= \sum_{j<k-1} (-)^{k-1-j}(\psi'_{i,k})^{(k-2-j)}(\tau_i)p^{(j)}(\tau_i) \\
&= \sum_{j<k-1} (-)^{k-1-j}[-\sum_{\ell=0}^{k-2} a_{\ell,k-1}\psi_{\ell+i,k-1}^{(k-2-j)}(\tau_i)p^{(j)}(\tau_i)] \\
&= \sum_{\ell=0}^{k-2} a_{\ell,k-1} \sum_{j<k-1} (-)^{k-2-j}\psi_{\ell+i,k-1}^{(k-2-j)}(\tau_i)p^{(j)}(\tau_i) \\
&= \sum_{\ell=0}^{k-2} a_{\ell,k-1}\, \lambda_{i+\ell,k-1}p \quad .
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} f(i)N_{i,k-1} = p = \sum_{i\in\mathbb{Z}} (\lambda_{i,k}p)N_{i,k} &= \sum_{\ell=0}^{k-2} a_{\ell,k-1} \sum_{i\in\mathbb{Z}} (\lambda_{i+\ell,k-1}p)N_{i,k} \\
&= \sum_{\ell=0}^{k-2} a_{\ell,k-1} \sum_{i\in\mathbb{Z}} f(i+\ell)N_{i,k} = \sum_{\ell=0}^{k-2} a_{\ell,k-1} \sum_{i\in\mathbb{Z}} f(i)N_{i,k}(\cdot+\ell) \quad .
\end{aligned}$$

We have proved

<u>Lemma 1.</u> <u>For any polynomial of degree</u> $\le k-2$

$$\sum_{i\in\mathbb{Z}} f(i)N_{i,k-1} = \sum_{i\in\mathbb{Z}} f(i)(\sum_{\ell=0}^{k-2} a_{\ell,k-1}N_{i,k}(\cdot+\ell)) \quad .$$

3. Box splines on a three-direction mesh.

As defined in [BH 1], the box spline M_Ξ is the distribution on $\mathbb{R}^m$ given by the rule:

$$M_\Xi : \phi \mapsto \int_{[0,1]^n} \phi\left(\sum_{i=1}^{n} \lambda(i)\xi_i\right)d\lambda$$

for some sequence $\Xi := (\xi_i)_1^n$ in $\mathbb{R}^m$. In our case, $m = 2$. Let e_i be the unit vector along x_i-axis $(i=1,2)$, and

$$d_1 := e_1,\ d_2 := e_1+e_2,\ d_3 := e_2 \quad .$$

For positive integers r, s and t, let $\Xi = (\xi_i)_1^{r+s+t}$ be the sequence in $\mathbb{R}^2$ given by

$$\xi_1 = \cdots = \xi_r = d_1,\ \xi_{r+1} = \cdots = \xi_{r+s} = d_2 \ \text{ and } \ \xi_{r+s+1} = \cdots = \xi_{r+s+t} = d_3 \quad .$$

From now we will write $M_{r,s,t}$ instead of M_Ξ. Caution! Our notation is slightly different from [BH 3]. In [BH 3], $d_2 = e_2$ and $d_3 = e_1+e_2$. Thus our $M_{r,s,t}$ is just $M_{r,t,s}$ in the sense given by [BH 3].

The smoothness of $M_{r,s,t}$ depends on the direction multiplicities. From [BH 3] we have

$$M_{r,s,t} \in L_\infty^{(d)} \subseteq C^{(d-1)}$$

with $d = \min\{r+s,s+t,t+r\} - 1$.

Now we define

$$B_{r,s,t}(x_1,x_2) := \sum_{\lambda_1=0}^{0} \cdots \sum_{\lambda_{r-1}=0}^{r-2} \sum_{\mu_1=0}^{0} \cdots \sum_{\mu_{s-1}=0}^{s-2} \sum_{\nu_1=0}^{0} \cdots \sum_{\nu_{t-1}=0}^{t-2}$$

$$(a_{\lambda_1,1} \cdots a_{\lambda_{r-1},r-1} a_{\mu_1,1} \cdots a_{\mu_{s-1},s-1} a_{\nu_1,1} \cdots a_{\nu_{t-1},t-1} \times \tag{3}$$

$$M_{r,s,t}(x_1+\lambda_1+\cdots+\lambda_{r-1}+\mu_1+\cdots+\mu_{s-1}, x_2+\mu_1+\cdots+\mu_{s-1}+\nu_1+\cdots+\nu_{t-1})$$

$$\text{for } (r,s,t) \text{ with } \min\{r,s,t\} > 1 \quad ,$$

where a has the meaning determined by (1).

The reason for introducing $B_{r,s,t}$ will be clear after we prove the following

<u>Lemma 2.</u> <u>For any bivariate polynomial of degree</u> $\leq r+s+t-2$, <u>we have</u>

$$1^{\circ} \quad D_1^r(D_1+D_2)^s[\sum_{j\in\mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r,s,t-1})(\cdot-j)] = 0 \quad ;$$

$$2^{\circ} \quad D_1^r D_2^t[\sum_{j\in\mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r,s-1,t})(\cdot-j)] = 0 \quad ;$$

$$3^{\circ} \quad (D_1+D_2)^s D_2^t[\sum_{j\in\mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r-1,s,t})(\cdot-j)] = 0 \quad .$$

<u>Here, as usual,</u> $j = (j_1,j_2) \in \mathbb{Z}^2$, $x = (x_1,x_2) \in \mathbb{R}^2$, $D_i = \frac{\partial}{\partial x_i}$, $\nabla_i f = f-f(\cdot-e_i)$ $(i = 1,2)$.

Proof. By symmetry, 2° and 3° follow from 1°. Thus we only need to prove 1°. Suppose

$$B_{r,s,t-1} = \sum_{i\in\mathbb{Z}^2} b_i M_{r,s,t-1}(\cdot+i) \quad .$$

Then, by the definition of $B_{r,s,t}$, we have

$$B_{r,s,t} = \sum_{i\in\mathbb{Z}^2} b_i \sum_{\ell=0}^{t-2} a_{\ell,t-1} M_{r,s,t}(\cdot+i+\ell e_2) \quad .$$

Hence

$$D_1^r(D_1+d_2)^s[\sum_{j\in\mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r,s,t-1})(\cdot-j)]$$

$$= \sum_i b_i \sum_{j\in\mathbb{Z}^2} (\nabla_1^r(\nabla_1+\nabla_2)^s p)(j)\{[\sum_{\ell=0}^{t-2} a_{\ell,t-1} M_{0,0,t}(\cdot-j+i+\ell e_2)] - M_{0,0,t-1}(\cdot-j+i)\} \quad .$$

For any test function ϕ, one can easily check that

$$\langle M_{0,0,t},\phi\rangle = \int M_t(x_2)\phi(0,x_2)dx_2 \quad .$$

Thus

$$\langle \sum_{j\in\mathbb{Z}^2} (\nabla_1^r(\nabla_1+\nabla_2)^s p)(j)\Big[\sum_{\ell=0}^{t-2} a_{\ell,t-1}M_{0,0,t}(\cdot-j+i+\ell e_2) - M_{0,0,t-1}(\cdot-j+i)\Big],\ \phi\rangle$$

$$= \int \sum_{j\in\mathbb{Z}^2} (\nabla_1^r(\nabla_1+\nabla_2)^s p)(j)\Big[\sum_{\ell=0}^{t-2} a_{\ell,t-1}M_t(x_2-j_2+i_2+\ell)-M_{t-1}(x_2-j_2+i_2)\Big]\phi(0,x_2)dx_2 = 0$$

by Lemma 1, since $\nabla_1^r(\nabla_1+\nabla_2)^s p$ has degree $< t-2$. This completes the proof of Lemma 2.

4. Quasi-interpolant scheme.

In the following, r, s and t are always integers. Let

$$I := \{(r,s,t) \mid r+s+t = 2\rho+4 \text{ and } 2 \le r,s,t \le \rho+1\}$$

$$J_1 := \{(r,s,t) \mid r+s+t = 2\rho+3 \text{ and } 2 \le r,s,t \le \rho+1\}$$

$$J_2 := \{(r,s,t) \mid r+s+t = 2\rho+3 \text{ and } 2 \le r,s,t \le \rho\}$$

$$K := \{(r,s,t) \mid r+s+t = 2\rho+2 \text{ and } 2 \le r,s,t \le \rho\} .$$

We have

$$I = \{(r,2,t)|r+2+t=2\rho+4, 2\le r,t\le\rho+1\} \cup \{(r,s,t)|r+s+t=2\rho+4, 3\le s\le\rho+1, 2\le r,t\le\rho+1\}$$

$$= \{(\rho+1,2,\rho+1)\} \cup \{(r,s,t)|r+s+t=2\rho+4, 3\le s\le\rho+1, 2\le r,t\le\rho\} \cup$$
$$\{(r,s,t)|r+s+t=2\rho+4, 3\le s\le\rho+1, 2\le r,t \text{ and } \max\{r,t\}=\rho+1\}$$

$$= \{(\rho+1,2,\rho+1)\} \cup \{(r,s,t)|r+s+t=2\rho+4, 4\le s\le\rho+1, 2\le r,t\le\rho\} \cup$$
$$\{(r,s,t)|r+s+t=2\rho+4, 3\le s\le\rho+1, 2\le r,t \text{ and } \max\{r,t\}=\rho+1\} . \tag{4}$$

Similarly,

$$J_1 = \{(r,s,t)|r+s+t=2\rho+3,\ 3\le s\le\rho+1 \text{ and } 2\le r,t\le\rho\} \cup$$
$$\{(r,s,t)|r+s+t=2\rho+3,\ 2\le s\le\rho,\ 2\le r,t \text{ and } \max\{r,t\} = \rho+1\} , \tag{5}$$

and

$$J_2 = \{(r,s,t)|r+s+t = 2\rho+3,\ 3 \le s \le \rho \text{ and } 2 \le r,t \le \rho\} . \tag{6}$$

Therefore

$$|I| \quad |J_2| - |J_1| + |K| = 1 . \tag{7}$$

Here, by $|E|$ we mean the cardinality of the set E.

In the following we use the convention that the empty sum has value 0.

Now we construct B as follows:

$$B := \sum_{(r,s,t)\in I} B_{r,s,t} - \sum_{(r,s,t)\in J_1} B_{r,s,t} \qquad (8)$$
$$- \sum_{(r,s,t)\in J_2} B_{r,s,t} + \sum_{(r,s,t)\in K} B_{r,s,t} .$$

<u>Lemma 3.</u> $\sum_{j\in\mathbb{Z}^2} B(\cdot-j) = 1.$

Proof. From [BH 1] we have

$$\sum_{j\in\mathbb{Z}^2} M_{r,s,t}(\cdot-j) = 1 .$$

Then (2) and (3) yield

$$\sum_{j\in\mathbb{Z}^2} B_{r,s,t}(\cdot-j) = 1 .$$

Therefore

$$\sum_{j\in\mathbb{Z}^2} B(\cdot-j) = |I| - |J_1| - |J_2| + |K| = 1 .$$

The following lemma plays an essential role in this paper.

<u>Lemma 4.</u> <u>For</u> $k = 2\rho+2$, $B \in \pi^{\rho}_{k,\Delta}$ <u>and</u>

$$p - \sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j) \quad \underline{\text{is a polynomial of degree}} < \deg p$$

<u>for any polynomial of degree</u> $\leq k-1$.

Proof. We first show that

$$D_1^{q_1} D_2^{q_2} [\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j)] \quad \text{is a constant for any} \quad (q_1,q_2) \in \mathbb{Z}^2 \qquad (9)$$
$$\text{with} \quad q_1 \geq 0,\ q_2 \geq 0 \quad \text{and} \quad q_1+q_2 = \deg p \leq k-1 .$$

Fix q_1 and q_2. Consider the following index sets:

$$E_1 := \{(r,s,t) \mid r > q_1 \text{ and } t > q_2\}$$

$$E_2 := \{(r,s,t) \mid r < q_1 \text{ and } t < q_2\}$$

$$E_3 := \{(r,s,t) \mid r < q_1 \text{ and } t > q_2\}$$

$$E_4 := \{(r,s,t) \mid r > q_1 \text{ and } t < q_2\} \ .$$

Then $\{E_i \ ; i = 1,2,3,4\}$ forms a partition of Z_+^3. To prove (10) it is sufficient to show that

$$D_1^{q_1} D_2^{q_2} \sum_{j \in Z^2} p(j) \Big(\sum_{(r,s,t)\in I\cap E_i} B_{r,s,t} - \sum_{(r,s,t)\in J_1\cap E_i} B_{r,s,t}$$

$$- \sum_{(r,s,t)\in J_2\cap E_i} B_{r,s,t} + \sum_{(r,s,t)\in K\cap E_i} B_{r,s,t} \Big)(\cdot - j)$$

is a constant for each $i = 1,2,3$ or 4. Thus we have to split our consideration into the four cases: $i = 1,2,3$ or 4.

Case $i = 1$. Then $r > q_1$, and $t > q_2$. We have

$$D_1^{q_1} D_2^{q_2} \sum_j p(j) M_{r,s,t}(\cdot - j) = \sum_j (\nabla_1^{q_1} \nabla_2^{q_2} p)(j) M_{r-q_1,s,t-q_2}(\cdot - j) \ ,$$

which is a constant, since $\nabla_1^{q_1} \nabla_2^{q_2} p$ is a constant. It follows that

$$D_1^{q_1} D_2^{q_2} \sum_j p(j) B_{r,s,t}(\cdot - j) \text{ is a constant for any } (r,s,t)$$

$$\text{with } r > q_1 \text{ and } t > q_2 \ .$$

Hence

$$D_1^{q_1} D_2^{q_2} \sum_{j \in Z^2} p(j) \Big(\sum_{(r,s,t)\in I\cap E_1} B_{r,s,t} - \sum_{(r,s,t)\in J_1\cap E_1} B_{r,s,t}$$

$$- \sum_{(r,s,t)\in J_2\cap E_1} B_{r,s,t} + \sum_{(r,s,t)\in K\cap E_1} B_{r,s,t} \Big)(\cdot - j)$$

is a constant.

Case $i = 2$. In this case, $r \leq q_1$ and $t \leq q_2$. Note that

$$(\rho+1,2,\rho+1) \notin E_2 \quad .$$

This is true, because $r \leq q_1$ and $t \leq q_2$ imply that

$$r+t \leq q_1+q_2 \leq 2\rho+1 \quad .$$

Now (4), (5) and (6) tell us that

$$\sum_{(r,s,t)\in I\cap E_2} B_{r,s,t} - \sum_{(r,s,t)\in J_1\cap E_2} B_{r,s,t} - \sum_{(r,s,t)\in J_2\cap E_2} B_{r,s,t} +$$

$$\sum_{(r,s,t)\in K\cap E_2} B_{r,s,t} = \sum_{s=4}^{\rho+1} \sum_{\substack{2\leq r,t\leq\rho \\ r+t=2\rho+4-s \\ r\leq q_1,\, t\leq q_2}} (B_{r,s,t} - B_{r,s-1,t}) +$$

$$\sum_{s=3}^{\rho+1} \sum_{\substack{2\leq r,t \\ \max\{r,t\}=\rho+1 \\ r+t=2\rho+4-s \\ r\leq q_1,\, t\leq q_2}} (B_{r,s,t}-B_{r,s-1,t}) - \sum_{s=3}^{\rho+1} \sum_{\substack{2\leq r,t\leq\rho \\ r+t=2\rho+3-s \\ r\leq q_1,\, t\leq q_2}} (B_{r,s,t}-B_{r,s-1,t}) \quad , \tag{10}$$

while

$$D_1^{q_1} D_2^{q_2} [\sum_{j\in\mathbb{Z}^2} p(j)(B_{r,s,t}-B_{r,s-1,t})(\cdot-j)] =$$

$$D_1^{q_1-r} D_2^{q_2-t} \{D_1^r D_2^t [\sum_{j\in\mathbb{Z}^2} p(j)(B_{r,s,t}-B_{r,s-1,t})(\cdot-j)]\} = 0$$

by Lemma 2. Therefore

$$D_1^{q_1} D_2^{q_2} [\sum_{j\in\mathbb{Z}^2} p(j)(\sum_{(r,s,t)\in I\cap E_2} B_{r,s,t} - \sum_{(r,s,t)\in J_1\cap E_2} B_{r,s,t}$$

$$- \sum_{(r,s,t)\in J_2\cap E_2} B_{r,s,t} + \sum_{(r,s,t)\in K\cap E_2} B_{r,s,t})(\cdot-j)] = 0 \quad .$$

Case $i = 3$. Then $r \leq q_1$ and $t > q_2$. We have

$$D_1^{q_1} D_2^{q_2} = D_1^r D_1^{q_1-r} D_2^{q_2} = D_1^r (D_1+D_2-D_2)^{q_1-r} D_2^{q_2}$$

$$= D_1^r [\sum_{\ell=0}^{q_1-r} (-1)^{q_1-r-\ell} (D_1+D_2)^{\ell} D_2^{q_1-r-\ell} \binom{q_1-r}{\ell}] D_2^{q_2}$$

$$= D_1^r [\sum_{\ell=0}^{q_1-r} (-1)^{q_1-r-\ell} \binom{q_1-r}{\ell} (D_1+D_2)^{\ell} D_2^{q_1+q_2-r-\ell}]$$

$$= D_1^r [(\sum_{\ell=0}^{q_1+q_2-r-t} + \sum_{\ell=q_1+q_2-r-t+1}^{s-1} + \sum_{\ell=s}^{q_1-r}) \tag{11}$$

$$(-1)^{q_1-r-\ell} \binom{q_1-r}{\ell} (D_1+D_2)^{\ell} D_2^{q_1+q_2-r-\ell}]$$

$$= D_1^r (D_1+D_2)^s H_{r,s} + D_1^r D_2^t G_{r,t} +$$

$$D_1^r [\sum_{\ell=q_1+q_2-r-t+1}^{s-1} (-1)^{q_1-r-\ell} \binom{q_1-r}{\ell} (D_1+D_2)^{\ell} D_2^{q_1+q_2-r-\ell}] \quad ,$$

where the differential operators $H_{r,s}$ and $G_{r,t}$ are defined as follows:

$$H_{r,s} := \sum_{\ell=s}^{q_1-r} (-1)^{q_1-r-\ell} \binom{q_1-r}{\ell} (D_1+D_2)^{\ell-s} D_2^{q_1+q_2-r-\ell}$$

$$G_{r,t} := \sum_{\ell=0}^{q_1+q_2-r-t} (-1)^{q_1-r-\ell} \binom{q_1-r}{\ell} (D_1+D_2)^{\ell} D_2^{q_1+q_2-r-t-\ell} \quad .$$

For the third term in (11) we observe that

$$D_1^r (D_1+D_2)^{\ell} D_2^{q_1+q_2-r-\ell} (\sum_{j \in \mathbb{Z}^2} p(j) M_{r,s,t}(\cdot - j))$$

$$= \sum_{j \in \mathbb{Z}^2} (\nabla_1^r (\nabla_1+\nabla_2)^{\ell} \nabla_2^{q_1+q_2-r-\ell} p)(j) M_{0,s-\ell,t-(q_1+q_2-r-\ell)}(\cdot - j)$$

is a constant for $\ell \in [q_1+q_2-r-t+1, s-1]$, because

$$s-\ell > 0 \quad \text{and} \quad t-(q_1+q_2-r-\ell) > 0 \quad .$$

Thus we can omit the third term of (11) in the following discussion. Now we want to prove that

$$\sum_{j\in\mathbb{Z}^2} p(j)\Big(\sum_{I\cap E_3} - \sum_{J_1\cap E_3} - \sum_{J_2\cap E_3} + \sum_{K\cap E_3}\Big)[H_{r,s}D_1^r(D_1+D_2)^s + G_{r,t}D_1^rD_2^t]B_{r,s,t}(\cdot-j)$$

is a constant. Let

$$U := \sum_{j\in\mathbb{Z}^2} p(j)\Big(\sum_{I\cap E_3} - \sum_{J_1\cap E_3} - \sum_{J_2\cap E_3} + \sum_{K\cap E_3}\Big)(H_{r,s}D_1^r(D_1+D_2)^3B_{r,s,t}(\cdot-j))$$

$$V := \sum_{j\in\mathbb{Z}^2} p(j)\Big(\sum_{I\cap E_3} - \sum_{J_1\cap E_3} - \sum_{J_2\cap E_3} + \sum_{K\cap E_3}\Big)(G_{r,t}D_1^rD_2^t)B_{r,s,t}(\cdot-j)) \quad .$$

Then we can argue separately for U and V. Interchange s and t in (4), (5) and (6). We can write down (cf. (10))

$$\begin{aligned}
U = {} & \sum_{j\in\mathbb{Z}^2} p(j)H_{\rho+1,\rho+1}\, D_1^{\rho+1}(D_1+D_2)^{\rho+1}B_{\rho+1,\rho+1,2}(\cdot-j) \\
& + \sum_{j\in\mathbb{Z}^2} p(j) \sum_{t=4}^{\rho+1} \sum_{\substack{3\le r,s\le\rho \\ r+s=2\rho+4-t \\ r\le q_1,\, t\ge q_2}} H_{r,s}D_1^r(D_1+D_2)^s(B_{r,s,t} - B_{r,s,t-1})(\cdot-j) \\
& + \sum_{j\in\mathbb{Z}^2} p(j) \sum_{t=3}^{\rho+1} \sum_{\substack{2\le r,s \\ \max\{r,s\}=\rho+1 \\ r+s=2\rho+4-t \\ r\le q_1,\, t\ge q_2}} H_{r,s}D_1^r(D_1+D_2)^s(B_{r,s,t} - B_{r,s,t-1})(\cdot-j) \qquad (12) \\
& + \sum_{j\in\mathbb{Z}^2} p(j) \sum_{t=3}^{\rho+1} \sum_{\substack{2\le r,s\le\rho \\ r+s=2\rho+3-t \\ r\le q_1,\, t\ge q_2}} H_{r,s}D_1^r(D_1+D_2)^s(B_{r,s,t} - B_{r,s,t-1})(\cdot-j) \quad .
\end{aligned}$$

However, $H_{\rho+1,\rho+1} = 0$, because $q_1 \le 2\rho+1 < (\rho+1) + (\rho+1)$. For the second term of the above expression, we have

$$\sum_{j\in\mathbb{Z}^2} p(j) \sum_{t=4}^{\rho+1} \sum_{\substack{3\le r,s\le\rho \\ r+s=2\rho+4-t \\ r\le q_1,\, t>q_2}} H_{r,s} D_1^r (D_1+D_2)^s (B_{r,s,t} - B_{r,s,t-1})(\cdot-j)$$

$$= \sum_{t=4} \sum_{\substack{3\le r,s\le\rho \\ r+s=2\rho+4-t \\ r\le q_1,\, t>q_2}} H_{r,s} D_1^r (D_1+D_2)^s \Big[\sum_{j\in\mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r,s,t-1})(\cdot-j)\Big] = 0$$

by Lemma 2. The third and fourth terms of (12) are also zero by the same argument. Thus $U = 0$. Similarly we can show $V = 0$. Therefore

$$D_1^{q_1} D_2^{q_2} \Big\{ \sum_{j\in\mathbb{Z}^2} p(j)\Big[\Big(\sum_{I\cap E_3} - \sum_{J_1\cap E_3} - \sum_{J_2\cap E_3} + \sum_{K\cap E_3} \Big) B_{r,s,t}(\cdot-j)\Big]\Big\}$$

is a constant.

Case $i = 4$. In this case $r > q_1$ and $t \le q_2$, and the argument is as in Case 3.

So far we have proved statement (9). Now

$$p - \sum_{j\in\mathbb{Z}^2} p(j) B(\cdot-j)$$

is a polynomial of degree $\le \deg p$. For (q_1, q_2) with $q_1 \ge 0$, $q_2 \ge 0$ and $q_1+q_2 = \deg p$ we have

$$\nabla_1^{q_1}\nabla_2^{q_2}\Big(p - \sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j)\Big) = \nabla_1^{q_1}\nabla_2^{q_2} p - \sum_j p(j)\big(\nabla_1^{q_1}\nabla_2^{q_2} B(\cdot-j)\big)$$

$$= \nabla_1^{q_1}\nabla_2^{q_2}\, p - \sum_j (\nabla_1^{q_1}\nabla_2^{q_2} p)(j) B(\cdot-j) \quad .$$

However, $\nabla_1^{q_1}\nabla_2^{q_2} p$ is a constant. Hence

$$\sum_j (\nabla_1^{q_1}\nabla_2^{q_2} p)(j) B(\cdot-j) = \nabla_1^{q_1}\nabla_2^{q_2} p$$

by Lemma 3. Therefore

$$\nabla_1^{q_1}\nabla_2^{q_2}\Big(p - \sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j)\Big) = 0, \text{ for any } (q_1,q_2) \text{ with } q_1 \geq 0, q_2 \geq 0 \text{ and } q_1+q_2 = \deg p .$$

This shows that $p - \sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j)$ is a polynomial of degree < deg p. Thus Lemma 4 is proved.

Now we can prove

Theorem 2. The mapping T defined by

$$T : p \mapsto \sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j) \quad , \quad p \in \pi_{k-1}$$

is one-to-one and onto π_{k-1}.

Proof. π_{k-1} is a linear space of finite dimension, and T is a linear mapping from π_{k-1} to π_{k-1} by Lemma 4. If $p \neq 0$, then $\deg p \geq 0$. Lemma 4 tells us that $\sum_j p(j)B(\cdot-j)$ has the same degree as p; that is $\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j) \neq 0$. This shows that T is one-to-one. Since π_{k-1} is finite-dimensional, T is also onto. The proof of Theorem 2 is complete.

Now combining Theorem 1 and Theorem 2 gives

Theorem 3. If $k = 2\rho+2$ and $S = \pi_{k,\Delta}^{\rho}$, then

$$\operatorname{dist}(f,S_h) = O(h^k)$$

for any sufficiently smooth function f.

Remark. From the above arguments we see that Theorem 3 remains true for $k \geq 2\rho+2$.

We show in Section 5 that the approximation order of $\pi_{4,\Delta}^{1}$ is 4. Thus, in general, Theorem 3 cannot be improved.

For the general case, we also have

Theorem 4. If $S = \pi_{k,\Delta}^{\rho}$ and $\rho \leq \rho(k) := \lfloor (2k-2)/3 \rfloor$, then

$$\operatorname{dist}(f,S_h) = O(h^{m(k)-2})$$

for any sufficiently smooth function f.

Proof. From [BH 1] we already know

$$\operatorname{dist}(f, S_h) = O(h^{\rho+2}) \quad .$$

If $2k \le 3\rho+4$, then

$$m(k) - 2 \le 2(k-\rho) - 2 = 2k - 2\rho-2 \le \rho+2 \quad .$$

Hence Theorem 4 holds for $2k \le 3\rho+4$. If $k \ge 2\rho+2$, then

$$m(k) - 2 \le k-1 \quad .$$

Thus Theorem 4 follows from Theorem 3. Now assume $2k \ge 3\rho+5$ and $k < 2\rho+2$.

Let

$$\sigma := 2\rho+2-k, \; k' := k-3\sigma, \; \rho' := \rho-2\sigma \quad .$$

Then

$$\rho' = \rho-2\sigma = \rho-2(2\rho+2-k) = 2k-3\rho-4 \ge 1 \quad ,$$

and

$$k' = k-3\sigma = 4k-6\rho-6 = 2(2k-3\rho-4) + 2 = 2\rho'+2 \quad .$$

Let

$$I' := \{(r,s,t) \mid r+s+t = 2\rho'+4 \text{ and } 2 \le r,s,t \le \rho'+1\}$$

$$J_1' := \{(r,s,t) \mid r+s+t = 2\rho'+3 \text{ and } 2 \le r,s,t \le \rho'+1\}$$

$$J_2' := \{(r,s,t) \mid r+s+t = 2\rho'+3 \text{ and } 2 \le r,s,t \le \rho'\}$$

$$K' := \{(r,s,t) \mid r+s+t = 2\rho'+2 \text{ and } 2 \le r,s,t \le \rho'\} \quad .$$

Define

$$\tilde{B} = \Big(\sum_{(r,s,t)\in I'} - \sum_{(r,s,t)\in J_1'} - \sum_{(r,s,t)\in J_2'} + \sum_{(r,s,t)\in K'} \Big) B_{r+\sigma,s+\sigma,t+\sigma} \quad .$$

Then $\tilde{B} \in \pi^{\rho}_{k,\Delta}$. An argument similar to that used for Lemma 4 shows that

$$p - \sum_{j\in\mathbb{Z}^2} p(j)\tilde{B}(\cdot-j)$$

is a polynomial of degree $<$ deg p for any polynomial p with

deg $p \le k'-1+2\sigma$. However,

$$k'-1+2\sigma = k-3\sigma-1 + 2\sigma = k-\sigma-1 = 2k-2\rho-3 \quad .$$

Thus the mapping

$$p \longmapsto \sum_{j\in\mathbb{Z}^2} p(j)\tilde{B}(\cdot - j)$$

is one-to-one and onto $\pi_{2k-2\rho-3}$. Now Theorem 1 gives the required result: For any sufficiently smooth function f,

$$\mathrm{dist}(f, S_h) = O(h^{2k-2\rho-2}) \quad .$$

This ends the proof of Theorem 4.

5. Approximation order from bivariate C^1-quartics

In this section we will show that for $S = \pi^1_{4,\Delta}$ and

$$f : \longmapsto x_1^2 x_2^3, \; x = (x_1, x_2) \in \mathbb{R}^2 \quad ,$$

there exists a positive constant such that

$$\mathrm{dist}(f, S_h) > \mathrm{const} \cdot h^4 \quad .$$

To this end we shall follow [BH 2] and discuss B-nets in the following.

Given a triangle τ with vertices U, V and W, we associate each point x with its barycentric coordinates, i.e. with (u,v,w) for which

$$x = uU + vV + wW, \quad \text{and} \quad u+v+w = 1 \quad .$$

Any polynomial p of degree $\leq n$ can be represented by

$$p = \sum_{i+j+k=n} b_{ijk}\phi_{ijk}$$

with

$$\phi_{ijk}(x) := \frac{n!}{i!j!k!} u^i v^j w^k \quad ,$$

where b_{ijk} are uniquely determined by p. This representation gives rise to a function

$$b : x_{ijk} \longmapsto b_{ijk}, \; x_{ijk} := (iU+jV+kW)/n \quad \text{and} \quad i+j+k = n \quad .$$

This function is called the B(ernstein or ezier)-net for p (with respect to τ). (See [BH 2].)

To a given function $f \in \pi^0_{4,\Delta}$ we associate a function b_f so that b_f is defined on

$$J_4 := (\mathbb{Z}/4)^2$$

and b_f agrees with the B-net for f on each triangle of Δ. Obviously, b_f is well defined. We also call b_f the B-net for f with respect to Δ.

Let us now introduce some linear functionals on $\pi^0_{4,\Delta}$. Define

$$\lambda_{i1}^{(m,n)} f := b_f(m+\frac{i-1}{4},n) + b_f(m+\frac{i}{4},n) - b_f(m+\frac{i-1}{4},n-\frac{1}{4}) - b_f(m+\frac{i}{4},n+\frac{1}{4}) \quad ,$$

$$\lambda_{i2}^{(m,n)} f := b_f(m,n+\frac{i-1}{4}) + b_f(m,n+\frac{i}{4}) - b_f(m-\frac{1}{4},n+\frac{i-1}{4}) - b_f(m+\frac{1}{4},n+\frac{i}{4}) \quad ,$$

$$\lambda_{i3}^{(m,n)} f := b_f(m+\frac{i-1}{4},n+\frac{i-1}{4}) + b_f(m+\frac{i}{4},n+\frac{i}{4}) - b_f(m+\frac{i-1}{4},n+\frac{i}{4}) -$$

$$b_f(m+\frac{i}{4},n+\frac{i}{4}),\ i = 1,2,3,4;\ m,n \in \mathbb{Z} \quad .$$

Let

$$\Lambda_{ij} := \{\lambda_{ij}^{(m,n)} \mid m,n \in \mathbb{Z}\} \quad , \quad i = 1,2,3,4;\ j = 1,2,3$$

and

$$\Lambda := \bigcup_{i=1}^{4} \bigcup_{j=1}^{3} \Lambda_{ij} \quad .$$

If $f \in \pi^1_{4,\Delta}$ then $\lambda b_f = 0$ for any $\lambda \in \Lambda$ (see [F] and [BH 2]).

We extend each $\lambda \in \Lambda$ to the continuous linear functional λI on $C(\mathbb{R}^2)$ with the aid of the local linear map I which associates f with the unique element If of $\pi^0_{4,\Delta}$ which agrees with f on J_4. Let T be the mapping $f \mapsto b_{If}$ for $f \in C(\mathbb{R}^2)$, and let T_j be the shift operator $f \mapsto f(\cdot+j)$. We have the following

<u>Lemma 5</u>. T <u>is a linear mapping and commutes with any shift</u> T_j, $j \in \mathbb{Z}^2$.

Proof. It is obvious that T is a linear mapping. To prove the second statement we first show that I commutes with any T_j. Indeed,

$$T_j(If)(i) = If(i+j) = f(i+j) \qquad \text{for any } i \in J_4 \quad ,$$

$$I(T_jf)(i) = I(f(\cdot+j))(i) = f(j+i) \quad \text{for any } i \in J_4 \quad .$$

This shows that $T_jI = IT_j$. Next, we have to show that the mapping

$$g \mapsto b_g \quad , \quad g \in \pi^0_{4,\Delta}$$

commutes with any T_j. Let τ be a triangle of Δ. Then

$$g|_\tau = \sum_{p+q+r=4} b_{pqr}\phi_{pqr} \quad .$$

It follows that

$$g(\cdot+j)|_{\tau+j} = \sum_{p+q+r} b_{pqr}\phi_{pqr} \quad .$$

Hence the mapping $g \mapsto b_g$ commutes with any shift. The Lemma is proved.

<u>Corollary</u>. <u>If</u> $f \in \pi_5^1$ <u>and</u> $\lambda \in \Lambda$, <u>then</u> λb_{If} <u>is invariant under translates.</u>

Proof. By Lemma 5

$$\lambda(b_{If}(\cdot+j) - b_{If}) = \lambda(TT_j - If) = \lambda(T(T_jf - f)) \quad .$$

However, $T_jf - f \in \pi_4$; hence $\lambda(T(T_jf - f)) = 0$. This shows that

$$\lambda b_{If}(\cdot+j) = \lambda b_{If} \quad \text{for any} \quad j \in \mathbb{Z}^2 \quad .$$

The Corollary is proved.

Now let

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \end{bmatrix} := \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

and

$$\mu_h := \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{i=1}^{4} \sum_{j=1}^{3} a_{ij}\lambda_{ij}^{(m,n)} I\sigma_{1/h} \quad \text{with} \quad h > 0 \quad \text{and} \quad N = \lfloor 1/h \rfloor \quad . \tag{13}$$

Since $S_h \subseteq \ker \mu_h$, we have

$$\operatorname{dist}(f,S_h) > \operatorname{dist}(f, \ker \mu_h) = |\mu_h f|/\|\mu_h\| \quad . \tag{14}$$

By the above Corollary

$$\mu_h f = N^2 \sum_{i=1}^{4} \sum_{j=1}^{3} \lambda_{ij}^{(0,0)} I\sigma_{1/h} f \quad \text{for} \quad f \in \pi_5 \quad .$$

For $f : x \mapsto x_1^2 x_2^3$ we have $\sigma_{1/h} f = h^5 f$. Hence

$$\mu_h f = h^5 N^2 \sum_{i=1}^{4} \sum_{j=1}^{3} \lambda_{ij}^{(0,0)} If \quad .$$

It is easy to verify that

$$\sum_{i=1}^{4}\sum_{j=1}^{3}\lambda_{ij}^{(0,0)}\,If$$

$$= -\ [b_{If}(0,\tfrac{3}{4}) - b_{If}(0,-\tfrac{1}{4})] + [b_{If}(\tfrac{1}{4},\tfrac{3}{4}) - b_{If}(\tfrac{1}{4},-\tfrac{1}{4})]$$

$$-\ [b_{If}(\tfrac{2}{4},\tfrac{3}{4}) - b_{If}(\tfrac{2}{4},-\tfrac{1}{4})] + [b_{If}(\tfrac{3}{4},\tfrac{3}{4}) - b_{If}(\tfrac{3}{4},-\tfrac{1}{4})]$$

$$+\ [b_{If}(1,0) - b_{If}(0,0)] - [b_{If}(1,\tfrac{1}{4}) - b_{If}(0,\tfrac{1}{4})]$$

$$+\ [b_{If}(\tfrac{3}{4},\tfrac{1}{4}) - b_{If}(-\tfrac{1}{4},\tfrac{1}{4})] - [b_{If}(\tfrac{3}{4},\tfrac{2}{4}) - b_{If}(-\tfrac{1}{4},\tfrac{2}{4})] \quad .$$

Let τ_1 be the triangle with vertices $U = (0,0)$. $V = (0,1)$ and $W = (1,1)$. Then

$$(x_1,x_2) = u(0,0) + v(0,1) + w(1,1) \quad \text{with} \quad u+v+w = 1 \quad .$$

It follows that

$$u = 1-x_2 \quad , \quad v = x_2-x_1 \quad \text{and} \quad w = x_1 \quad .$$

Hence

$$x_{ijk} = (iU + jV + kW)/4 = (\tfrac{k}{4}, \tfrac{j+k}{4})$$

and

$$\phi_{ijk} = \frac{4!}{i!j!k!}(1-x_2)^i x_1^j (x_2-x_1)^k \quad .$$

Thus

$$If|_{\tau_1} = \sum_{p=0}^{4}\sum_{q=0}^{p} b_{If}(\tfrac{p}{4},\tfrac{q}{4})\phi_{4-q,q-p,p} \quad .$$

By Lemma 5 we have

$$I(f(\cdot-e_2))|_{\tau_1} = \sum_{p=0}^{4}\sum_{q=0}^{p} b_{If}(\tfrac{p}{4},\tfrac{q}{4} - 1)\phi_{4-q,q-p,p} \quad .$$

Therefore,

$$\sum_{p=0}^{4}\sum_{q=0}^{p} [b_{If}(\tfrac{p}{4},\tfrac{q}{4}) - b_{If}(\tfrac{p}{4},\tfrac{q}{4} -1)]\phi_{4-q,q-p,p} = I(f - f(\cdot-e_2))|_{\tau_1} \quad .$$

On the other hand

$$(f - f(\cdot-e_2))(x_1,x_2) = x_1^2x_2^3 - x_1^2(x_2-1)^3 = x_1^2(3x_2^2 - 3x_2 + 1)$$

$$= x_1^4 + 2x_1^3(x_2-x_1) + x_1^2(x_2-x_1)^2 - x_1^3(1-x_2) - x_1^2(x_2-x_1)(1-x_2) + x_1^2(1-x_2)^2$$

$$= \phi_{0,0,4} + \frac{1}{2}\phi_{0,1,3} + \frac{1}{6}\phi_{0,2,2} - \frac{1}{4}\phi_{1,0,3} - \frac{1}{12}\phi_{1,1,2} + \frac{1}{6}\phi_{2,0,2} \quad .$$

This yields the following result:

$$b_{If}(0,\tfrac{3}{4}) - b_{If}(0,-\tfrac{1}{4}) = 0$$

$$b_{If}(\tfrac{1}{4},\tfrac{3}{4}) - b_{If}(\tfrac{1}{4},-\tfrac{1}{4}) = 0$$

$$b_{If}(\tfrac{2}{4},\tfrac{3}{4}) - b_{If}(\tfrac{2}{4},-\tfrac{1}{4}) = -\tfrac{1}{12}$$

$$b_{If}(\tfrac{3}{4},\tfrac{3}{4}) - b_{If}(\tfrac{3}{4},-\tfrac{1}{4}) = -\tfrac{1}{4} \;.$$

Now we consider another triangle τ_2 with vertices $U = (0,0)$, $V = (1,0)$ and $W = (1,1)$. Then

$$u = 1 - x_1,\ v = x_1 - x_2 \text{ and } w = x_2 \;;$$

$$x_{ijk} = (\tfrac{j}{4}, \tfrac{j+k}{4})$$

$$\phi_{ijk} = (1-x_1)^i (x_1 - x_2)^j x_2^k \;.$$

Moreover, we have

$$f(x_1,x_2) - f(x_1-1,x_2) = (2x_1-1)x_2^3 = -(1-x_1)x_2^3 + (x_1-x_2)x_2^3 + x_2^4$$

$$= -\tfrac{1}{4}\phi_{1,0,3} + \tfrac{1}{4}\phi_{0,1,3} + \phi_{0,0,4} \;.$$

It follows that

$$b_{If}(1,0) - b_{If}(0,0) = 0$$

$$b_{If}(1,\tfrac{1}{4}) - b_{If}(0,\tfrac{1}{4}) = 0$$

$$b_{If}(\tfrac{3}{4},\tfrac{1}{4}) - b_{If}(-\tfrac{1}{4},\tfrac{1}{4}) = 0$$

$$b_{If}(\tfrac{3}{4},\tfrac{2}{4}) - b_{If}(-\tfrac{1}{4},\tfrac{2}{4}) = 0 \;.$$

In conclusion we obtain

$$\sum_{i=1}^{4} \sum_{j=1}^{3} a_{ij}\, \lambda_{ij}^{(0,0)} If = \frac{1}{12} - \frac{1}{4} = -\frac{1}{6} \;. \qquad (15)$$

Thus

$$|\mu_h f| = \frac{1}{6} h^5 N^2 \geq \frac{1}{12} h^3 \quad \text{for} \quad h < \frac{1}{4} \quad . \tag{16}$$

Furthermore, we have, for any $g \in C(\mathbb{R}^2)$,

$$\mu_h g = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{i=1}^{4} \sum_{j=1}^{3} a_{ij} \lambda_{ij}^{(m,n)} I\sigma_{1/h} g$$

$$= \sum_{m=0}^{N-1} [b_{I\sigma_{1/h}g}(m,-\tfrac{1}{4}) - b_{I\sigma_{1/h}g}(m+\tfrac{1}{4},-\tfrac{1}{4}) + b_{I\sigma_{1/h}g}(m+\tfrac{2}{4},-\tfrac{1}{4}) - b_{I\sigma_{1/h}g}(m+\tfrac{3}{4},-\tfrac{1}{4})]$$

$$- \sum_{m=0}^{N-1} [b_{I\sigma_{1/h}g}(m,N-\tfrac{1}{4}) - b_{I\sigma_{1/h}g}(m+\tfrac{1}{4},N-\tfrac{1}{4}) + b_{I\sigma_{1/h}g}(m+\tfrac{2}{4},N-\tfrac{1}{4}) - b_{I\sigma_{1/h}g}(m+\tfrac{3}{4},N-\tfrac{1}{4})]$$

$$+ \sum_{n=0}^{N-1} [-b_{I\sigma_{1/h}g}(0,n) + b_{I\sigma_{1/h}g}(0,n+\tfrac{1}{4}) - b_{I\sigma_{1/h}g}(-\tfrac{1}{4},n+\tfrac{1}{4}) + b_{I\sigma_{1/h}g}(-\tfrac{1}{4},n+\tfrac{2}{4})] \tag{17}$$

$$- \sum_{n=0}^{N-1} [-b_{I\sigma_{1/h}g}(N,n) + b_{I\sigma_{1/h}g}(N,n+\tfrac{1}{4}) - b_{I\sigma_{1/h}g}(N-\tfrac{1}{4},n+\tfrac{1}{4}) + b_{I\sigma_{1/h}g}(N-\tfrac{1}{4},n+\tfrac{2}{4})] \quad .$$

It is easily seen that

$$|b_{I\sigma_{1/h}g}| \leq \text{const} \quad g_C$$

where the const is independent of h. Hence (17) implies that

$$|\mu_h g| \leq 4N \text{ const} \quad g_C \leq \text{const} \cdot \frac{1}{h} \quad g_C \quad .$$

This shows that

$$\mu_h = 0(\frac{1}{h}) \quad . \tag{18}$$

Now (14), (16) and (18) yield the desired result:

$$\text{dist}(f, S_h) \geq \text{const } h^4$$

for some positive constant and the function $f : x \mapsto x_1^2 x_2^3$.

Acknowledgement

The author wishes to thank Professor C. de Boor and K. Höllig for their encouragement and several useful discussions.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER #2494	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Approximation by Smooth Bivariate Splines on a Three-Direction Mesh		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Rong-qing Jia		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE March 1983 13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

B-splines, bivariate, degree of approximation, pp, quasi-interpolants, linear functionals, smooth, spline functions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $S := \pi^{\rho}_{k,\Delta}$ be the space of bivariate piecewise polynomial functions in C^{ρ}, of degree $\leq k$, on the mesh Δ obtained from a uniform square mesh by drawing in the same diagonal in each square.

de Boor and Höllig have given the following upper bound

$$m \leq m(k) := \min\{2(k-\rho), k+1\}$$

for the approximation order m of S.

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